$$\frac{\partial \overline{K}_1}{\partial \beta_1} = -\frac{\partial \overline{K}_1}{\partial \beta_2} = \frac{P}{mG_0^2} \sqrt{G_0^2 - L_0^2}\,[(L_0 + G_0)\, x_c \cos(\beta_1 - \beta_2) - \tag{7}$$

$$(L_0 - G_0)\, y_c \sin(\beta_1 - \beta_2)] + \frac{3P\,(G_0^2 - L_0^2)}{8mR}\,(G_0 - L_0)^2 (A - B) \sin 2(\beta_1 - \beta_2) = 0$$

It is obvious that condition (7) is fulfilled for a suitable choice of arbitrary constants and, consequently, in the case being considered there also exists a family of periodic solutions (but with a lesser number of arbitrary constants).

We note that in the limit as $R \to \infty$ we arrive at the classical problem of the motion of a rigid body in a homogeneous gravity field, for which periodic solutions are obtained in the first two cases for sufficiently small $A - B$, x_c, y_c, z_c As a more detailed investigation shows, the last two cases lead to periodic solutions inherent only for the de Brun field and the central Newtonian field.

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ON THE ADMISSIBILITY OF APPLICATION OF PRECESSION EQUATIONS OF GYROSCOPIC SYSTEMS

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Precession equations are widely applied in gyroscopic systems. The conditions, under whose fulfillment the application of these equations is in a certain sense legitimate, have been established for linear autonomous systems and for certain special cases of nonlinear systems [1 – 3]. We give below the proof of precession theory for a wide class of nonlinear and nonautonomous systems.

We consider a system under the action of gyroscopic forces depending on a large positive parameter H, resistance forces with total dissipation, and other generalized forces $Q_k(q, t)$ depending on the coordinates q and on time t. Among the generalized forces $Q_k(q, t)$ there can occur potential, position-nonconservative (radial-correction), and other forces depending on the coordinates, perturbing forces depending explicitly on time, inertia forces, etc.

We shall write the equations of motion in the following form [1, 2]:

$$\frac{d}{dt}\frac{\partial T_2}{\partial q^{\bullet}_k} - \frac{\partial T_2}{\partial q_k} = Q_k(q, t) - \sum_{j=1}^{s}(b_{kj} + Hg_{kj})\, q_j^{\bullet} \qquad (k = 1, \ldots, s) \tag{1}$$

$$T_2 = \frac{1}{2}\sum_{k=1}^{s}\sum_{j=1}^{s} a_{kj} q_k^{\bullet} q_j^{\bullet}$$

Here T_2 is a positive-definite quadratic form in the generalized velocities $q^{\bullet}$. We assume that the inretia coefficients $a_{kj} = a_{jk}$, the coefficients $b_{kj} = b_{jk}$ of the dissipative forces, and the gyroscopic coefficients $g_{kj} = -g_{jk}$, are continuous functions of coordinates q and of time t. Many problems on the motion of nonlinear gyroscopic devices on a moving base are reduced to equations of form (1).

By substituting the value of T_2 into (1) we obtain the equivalent vector-matrix equation

$$\frac{d}{dt} A\mathbf{q}^{\bullet} = \frac{1}{2}\frac{\partial A}{\partial \mathbf{q}}\mathbf{q}^{\bullet}\cdot\mathbf{q}^{\bullet} + \mathbf{Q} - (B + HG)\,\mathbf{q}^{\bullet} \tag{2}$$

$$A = \| a_{kj} \|, \quad B = \| b_{kj} \|, \quad G = \| g_{kj} \|$$

$$\mathbf{Q} = (Q_1, \ldots, Q_s), \qquad \mathbf{q} = (q_1, \ldots, q_s)$$

Here, by the first term on the right-hand side we mean a vector whose projections are determined by the equalities

$$\frac{1}{2}\left(\frac{\partial A}{\partial \mathbf{q}}\mathbf{q}^{\bullet}\cdot\mathbf{q}^{\bullet}\right)_k = \frac{1}{2}\frac{\partial A}{\partial q_k}\mathbf{q}^{\bullet}\cdot\mathbf{q}^{\bullet}$$

In the Hamilton variables $\mathbf{q}$, $\mathbf{p} = A\mathbf{q}^{\bullet}$ Eq. (2) is equivalent to two first-order equations ($\mu = H^{-1}$ is a small parameter)

$$\mathbf{q}^{\bullet} = A^{-1}\mathbf{p} \tag{3}$$

$$\mu\mathbf{p}^{\bullet} = \frac{1}{2}\mu\,\frac{\partial A}{\partial \mathbf{q}}\, A^{-1}\mathbf{p}\cdot A^{-1}\mathbf{p} + \mu\mathbf{Q} - (\mu B + G)\, A^{-1}\mathbf{p}$$

These equations are a particular case of the following singularly-perturbed system:

$$\mathbf{q}^{\bullet} = \mathbf{f}(\mathbf{q}, \mathbf{p}, t, \mu), \qquad \mu\mathbf{p}^{\bullet} = \mathbf{F}(\mathbf{q}, \mathbf{p}, t, \mu) \tag{4}$$

whose truncated equations have the form:

$$\mathbf{u}^{\bullet} = \mathbf{f}(\mathbf{u}, \mathbf{v}, t, \mu), \qquad \mathbf{F}(\mathbf{u}, \mathbf{v}, t, \mu) = 0 \tag{5}$$

(in the truncated equations the vectors $\mathbf{u}$, $\mathbf{v}$ correspond to the vectors $\mathbf{q}$, $\mathbf{p}$).

For the case when the right-hand sides of Eqs. (4) do not depend on the small parameter μ, Tikhonov [4] proved a theorem according to which the solution $\mathbf{q}(t, \mu)$, $\mathbf{p}(t, \mu)$ of the full Eqs. (4) tends, when a number of conditions are fulfilled, as $\mu \to 0$ to the solution $\mathbf{u}(t)$, $\mathbf{v}(t)$ of the truncated system (5). Wasow [5] noted that all the results are applicable also to the case when the right-hand sides of Eqs. (4) depend on the small parameter μ in a sufficiently regular manner. But it is necessary to consider that this remark is valid when the following additional conditions, following directly from the proof of Tikhonov theorem, are fulfilled: (1) the function $\mathbf{F}(\mathbf{q}, \mathbf{p}, t, \mu)$ must be of zero order relative to parameter μ; (2) the equation $\mathbf{F}(\mathbf{q}, \mathbf{p}, t, \mu) = 0$ must have an isolated root $\mathbf{p} = \boldsymbol{\varphi}(\mathbf{q}, t, \mu)$ when $\mu = 0$.

We return to Eqs. (3). It is obvious that the first additional condition is fulfilled, while the second requires that the matrix G made up from the gyroscopic coefficients be nonsingular (hence it follows, in particular, that the number of coordinates should be even). Let us show that under the assumptions made and when $\det G \neq 0$, all the hypotheses of Tikhonov theorem are satisfied.

We set up the boundary layer equation

$$\frac{d\mathbf{p}}{d\tau} = \frac{1}{2}\mu\frac{\partial A}{\partial \mathbf{q}}A^{-1}\mathbf{p}\cdot A^{-1}\mathbf{p} + \mu\mathbf{Q} - (\mu B + G)A^{-1}\mathbf{p} \tag{6}$$

Here $\mathbf{q}$ and t are fixed, while the independent variable is τ. We equate the right-hand side of Eq. (6) to zero and we present the equality obtained in the following form:

$$(\mu B + G)A^{-1}\mathbf{p} = \mu\mathbf{Q} + \frac{1}{2}\mu\frac{\partial A}{\partial \mathbf{q}}A^{-1}\mathbf{p}\cdot A^{-1}\mathbf{p} \tag{7}$$

We seek the solution of this equation by the method of successive approximations. As the zero approximation we take $\mathbf{p}_0 = \boldsymbol{\varphi}_0 = 0$. We substitute this value into the right-hand side of equality (7) and we find the first approximation

$$\mathbf{p}_1 = \boldsymbol{\varphi}_1 = \mu\,[(\mu B + G)A^{-1}]^{-1}\mathbf{Q}$$

(in [2] it was proved that under total dissipation the matrix within the brackets is nonsingular). It is obvious that the second approximation $\mathbf{p}_2 = \boldsymbol{\varphi}_2$ differes from $\boldsymbol{\varphi}_1$ by a term containing μ^3. Therefore, for a sufficiently small μ the singular point $\boldsymbol{\varphi}$ of the boundary layer equation (6) is determined by the equality

$$\boldsymbol{\varphi}(\mathbf{q}, t, \mu) = \mu\,[(\mu B + G)A^{-1}]^{-1}\mathbf{Q} + O(\mu^3) \tag{8}$$

Let us investigate the stability of the singular point of the boundary layer for fixed $\mathbf{q}$ and t. To do this we set $\mathbf{p} = \boldsymbol{\varphi} + \mathbf{z}$, we replace $\mathbf{p}$ by this expression in Eq. (6), and we take into account that $\boldsymbol{\varphi}$ is a root of the right-hand side. We then obtain

$$\begin{aligned}\frac{d\mathbf{z}}{d\tau} = -\Big[(\mu B + G)A^{-1}\mathbf{z} - \frac{1}{2}\mu\frac{\partial A}{\partial \mathbf{q}}A^{-1}\boldsymbol{\varphi}\cdot A^{-1}\mathbf{z} -\\ \frac{1}{2}\mu\frac{\partial A}{\partial \mathbf{q}}A^{-1}\mathbf{z}\cdot A^{-1}\boldsymbol{\varphi}\Big] + \frac{1}{2}\mu\frac{\partial A}{\partial \mathbf{q}}A^{-1}\mathbf{z}\cdot A^{-1}\mathbf{z}\end{aligned} \tag{9}$$

For fixed $\mathbf{q}$ and t the coefficients of $\mathbf{z}$ should be treated as constant matrices. The singular point is of the order μ; therefore, for a sufficiently small value of μ the first-approximation equation takes the form

$$A\frac{d\mathbf{x}}{d\tau} + (\mu B + G)\mathbf{x} = 0 \qquad (A^{-1}\mathbf{z} = \mathbf{x})$$

In [2] it was shown that all roots of the corresponding characteristic equation have negative real parts (under the assumption that matrices A and B are positive definite). On the basis of Liapunov stability theorem, from the first-approximation equations we conclude that for sufficiently small values of $|\mathbf{z}|$ the trivial solution $\mathbf{z} = 0$ of the nonlinear equation (9), and, together with it, the singular point $\mathbf{p} = \boldsymbol{\varphi}(\mathbf{q}, t, \mu)$ of the boundary layer equation (6) with fixed $\mathbf{q}$ and t is asymptotically stable. From this it follows that Tikhonov theorem is applicable and the solution of the full Eqs. (3) is determined sufficiently accurately by the solution of the truncated system

$$\mathbf{u}^{\bullet} = A^{-1}\mathbf{v}, \quad \frac{1}{2}\mu\frac{\partial A}{\partial \mathbf{u}}A^{-1}\mathbf{v}\cdot A^{-1}\mathbf{v} + \mu\mathbf{Q} - (\mu B + G)A^{-1}\mathbf{v} = 0$$

On the basis of equality (8) the first term in the second equation can be discarded, as a result of which the truncated system takes the form

$$\mathbf{u}^{\cdot} = A^{-1}\mathbf{v}, \qquad \mu\mathbf{Q} - (\mu B + G)\, A^{-1}\mathbf{v} = 0$$

or, if we once again pass to the Lagrange's variables and replace μ by H^{-1},

$$(B + HG)\,\mathbf{u}^{\cdot} = \mathbf{Q}(\mathbf{u}, t) \tag{10}$$

which is a vector-matrix form of the precession equations (they are obtained from Eqs. (1) when $T_2 = 0$). Thus, we have proved the following theorem.

Theorem. If on a material system there act gyroscopic forces depending on a large parameter H, resistance forces with total dissipation, and other generalized forces $Q_k(q, t)$, then under the condition that the determinant of the matrix $G(q, t)$ composed of the gyroscopic coefficients is nonzero, the solution of the full nonlinear and nonautonomous differential equations (1) coincides sufficiently accurate with the solution of the precession equations. (Of course, it is assumed that the initial values of the vector $\mathbf{p}_0 = A(\mathbf{q}_0, t_0)\,\mathbf{q}_0^{\cdot}$ take part in the influence of the root $\mathbf{p} = \varphi(\mathbf{q}, t, \mu)$ of the boundary layer equation (7)).

This theorem can be given a simple geometric interpretation. The precession equation (10) is equivalent to the following two equations:

$$\mathbf{u}^{\cdot} = \mathbf{v}, \qquad (\mu B + G)\,\mathbf{v} = \mathbf{Q}(\mathbf{u}, t)$$

In the $2s$-dimensional phase space $(\mathbf{u} = \mathbf{q};\, \mathbf{v} = \mathbf{q}^{\cdot}) = (q_1, \ldots, q_s;\, q_1^{\cdot}, \ldots, q_s^{\cdot})$ the second equation defines an integral curve γ (the curve γ for nonautonomous systems deforms and shifts around in the space with time t) along which the image point $N(\mathbf{u}, \mathbf{v})$ of the precession equations moves. When the theorem's hypotheses are satisfied the image point $M(\mathbf{q}, \mathbf{q}^{\cdot})$ of the full equations of motion (1) rapidly approaches the curve γ and then moves along it.

We note that at the expense of decreasing the parameter μ the difference $\mathbf{q}(t, \mu) - \mathbf{u}(t, \mu)$ can be made as small as desired not for all $t > 0$ but only in a certain interval $(0, T)$. If we neglect the difference in the velocities of the motion of the image points M and N along curve γ, then we can consider that the solution $\mathbf{u}(t, \mu)$ of the precession equations is acceptable for all $t \geqslant 0$. The theorem proved on the acceptability of the solutions of precession equations for nonlinear and nonautonomous systems requires the presence of resistance with total dissipation. Therefore, it cannot replace the analogous theorems proved in [1, 2] for linear autonomous systems not containing resistance forces.

Let us show what the violation of the condition $\det G \neq 0$ leads to when all other conditions are fulfilled. Let the full system of equations be

$$\begin{aligned} \alpha^{\cdot\cdot} + k\alpha^{\cdot} + Hg_1\beta^{\cdot} + Hg_2\gamma^{\cdot} &= 0 \\ \beta^{\cdot\cdot} + k\beta^{\cdot} - Hg_1\alpha^{\cdot} + Hg_3\gamma^{\cdot} &= 0 \\ \gamma^{\cdot\cdot} + k\gamma^{\cdot} - Hg_2\alpha^{\cdot} - Hg_3\beta^{\cdot} &= 0 \end{aligned} \tag{11}$$

On the system act the resistance forces $-k\alpha^{\cdot}$, $-k\beta^{\cdot}$, $-k\gamma^{\cdot}$ with total dissipation and gyroscopic forces with the matrix

$$G = \begin{Vmatrix} 0 & g_1 & g_2 \\ -g_1 & 0 & g_3 \\ -g_2 & -g_3 & 0 \end{Vmatrix}$$

The precession equations (u, v, w correspond to α, β, γ) are

$$\begin{aligned} &ku^{\cdot} + Hg_1v^{\cdot} + Hg_2w^{\cdot} = 0, \\ &kv^{\cdot} - Hg_1u^{\cdot} + Hg_3w^{\cdot} = 0, \\ &kw^{\cdot} - Hg_2u^{\cdot} - Hg_3v^{\cdot} = 0 \end{aligned} \tag{12}$$

and for $k \neq 0$ and $\mu = H^{-1} \neq 0$ have the unique solution

$$u = \alpha_0, \quad v = \beta_0, \quad w = \gamma_0 \tag{13}$$

If we divide Eqs. (11) by H, introduce the small parameter $\mu = H^{-1}$, integrate them, and retain only the principal terms in the general solution, then we have

$$\alpha = \alpha_0 + g_3E, \ \beta = \beta_0 - g_2E, \ \gamma = \gamma_0 + g_1E$$

where

$$E = \frac{g_3\alpha_0^{\cdot} - g_2\beta_0^{\cdot} + g_1\gamma_0^{\cdot}}{k(g_1^2 + g_2^2 + g_3^2)} (1 - e^{-kt})$$

This solution differs from solution (13) by terms nondepending on the small parameter $\mu = H^{-1}$; therefore, the passage from the full equations (11) to the precession equations (12) is inadmissible (in the example given $\det G = 0$).

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INVARIANTS OF MULTIDIMENSIONAL SYSTEMS WITH ONE RESONANCE RELATION

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The description of invariants generated in systems of ordinary equations by homeomorphisms of a neighborhood of a singular point is connected both with stability problems [1, 2] as well as with the broader problems of the topological, analytical (or formal) classification of such systems [3, 4]. If the eigenvalues of the system's linear part are related by only one resonance relation, a reduction to normal form [5] enables us to extend the results obtained in [6] to invariants of an nth-order system [7]. Namely, we have shown that the group of all analytic